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Nonlinear gravity–capillary surface waves in a slowly varying current

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The propagation of nonlinear gravity–capillary surface waves in a deep slowly varying current is investigated using the conservation equations in the eikonal approximation. Graphical comparisons are made between solutions of the wave-slope and wavenumber equations for infinitesimal waves and finite amplitude waves. Finite amplitude effects are shown to be weaker for small amplitude capillary waves than for gravity waves. The 'wave barrier' noted by Gargett & Hughes (1972) for infinitesimal gravity waves on a slowly varying current is seen to be removed by finite amplitude effects.

1. Introduction

In this note we examine the wave-slope and wavenumber changes that occur when a train of deep-water nonlinear gravity–capillary surface waves encounters a gradual time-independent variation in surface current. An analysis of this problem for gravity waves has recently been made by Crapper (1972), who used the method of Whitham (1965*a*, *b*) on a Lagrangian suggested by Lighthill (1967). The infinitesimal gravity wave case has been investigated by Longuet-Higgins & Stewart (1961), who used a perturbation approach, and by Whitham (1962), who used the conservation equations for mass, momentum, and energy in the eikonal approximation. The eikonal-conservation-equation method will be used in this analysis.

For simplicity we shall consider only one horizontal space dimension. An extension of the analysis to two horizontal dimensions is straightforward. The changes in wave slope ϵ and wavenumber k for a train of initially uniform waves moving into a region of gradually varying surface current can be found from the conditions that the frequency of the surface wave, the mean total mass flux and the mean total energy flux are constant; that is,

$$\omega = \text{const.}, \tag{1.1}$$

$$\bar{s} = \left\langle \int_{-d(x)}^{\eta(x,t)} \frac{\partial \Phi}{\partial x}\, dz \right\rangle = \text{const.}, \tag{1.2}$$

$$\bar{F} = -\left\langle \int_{-d(x)}^{\eta(x,t)} \frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial t}\, dz \right\rangle - T\left\langle \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t} \Big/ \left[1+\left(\frac{\partial n}{\partial x}\right)^2\right]^{\frac{1}{2}} \right\rangle = \text{const.}, \tag{1.3}$$

where Φ is the total velocity potential, $\eta(x,t)$ is the equation of the surface, $-d(x)$ is the depth of the current and angular brackets denote averaging over the phase of the surface wave; T is the surface tension per unit density. The gradual variation of the surface current implies that the total velocity potential is

$$\Phi = \phi + \Lambda, \tag{1.4}$$

where ϕ is the velocity potential for a surface wave and Λ is the velocity potential for the current

$$U(x,z) = \partial\Lambda/\partial x. \tag{1.5}$$

We assume that U has a negligible depth dependence within several wavelengths of the surface; horizontal variations in U are balanced by vertical currents which are negligible near the surface.† The mean total energy flux can then be written as

$$\overline{F} = \overline{F}_0 + \overline{F}_{ex}, \tag{1.6}$$

where
$$\overline{F}_0 = \tfrac{1}{2}S^2\left\langle \int_{-d(x)}^{\eta(x,t)} \frac{\partial\Phi}{\partial x}\,dz \right\rangle \tag{1.7}$$

and
$$\overline{F}_{ex} = -\left\langle \int_{-\infty}^{\eta(x,t)} \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}\,dz \right\rangle - U(x,0)\left\langle \int_{-\infty}^{\eta(x,t)} \frac{\partial\phi}{\partial t}\,dz \right\rangle - T\left\langle \frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial t}\Big/\left[1+\left(\frac{\partial\eta}{\partial x}\right)^2\right]^{\frac{1}{2}} \right\rangle, \tag{1.8}$$

where $-S$ is a constant current about which variations in $U(x,0)$ will be measured; the condition

$$\partial\Lambda/\partial t = -\tfrac{1}{2}S^2, \tag{1.9}$$

which follows from Bernoulli's equation, was used in (1.7). Since $\overline{F}_0$ is proportional to $\bar{s}$, we obtain

$$\overline{F}_{ex} = \text{const.} \tag{1.10}$$

This equation and the Doppler relation (Landau & Lifschitz 1959, §67),

$$ck + U(x,0)\,k = \text{const.}, \tag{1.11}$$

where c is the phase velocity of the surface wave, suffice to determine the variations in wave slope and wavenumber.

† These conditions can be realized when a steady stream flows over an irregular bed of slow variation and great depth (see Lamb 1932, art. 246). Internal wave motion can also produce the assumed current (see Defant 1960, chap. XVI).

2. Nonlinear waves

Calculation of (1.11) to second order in the wave slope ϵ and (1.10) to fourth order gives†

$$\omega = c_0 k[1+\tfrac{1}{2}\epsilon^2 a_2] + U(x,0)\,k = \text{const.}, \tag{2.1}$$

where

$$c_0 = (g/k+Tk)^{\frac{1}{2}}, \tag{2.2}$$

$$a_2 = 1+\frac{3}{2}\frac{k^2T}{g-2k^2T}-\frac{3}{8}\frac{k^2T}{g+k^2T}, \tag{2.3}$$

and

$$\frac{\overline{F}_{ex}}{\omega} = \tfrac{1}{2}\epsilon^2\left[\left(\frac{c_0}{k}\right)^2 f_1+\frac{T}{k}f_2+U(x,0)\frac{c_0}{k^2}f_3\right] = \text{const.}, \tag{2.4}$$

where

$$f_1 = \tfrac{1}{2}+\epsilon^2\left[1+\frac{15}{4}\frac{k^2T}{g-2k^2T}+\frac{9}{4}\frac{(k^2T)^2}{(g-2k^2T)^2}\right], \tag{2.5}$$

$$f_2 = 1+\epsilon^2\left[\frac{7}{8}+\frac{15}{2}\frac{k^2T}{g-2k^2T}+\frac{3}{8}\frac{k^2T}{g+k^2T}+9\frac{(k^2T)^2}{(g-2k^2T)^2}\right], \tag{2.6}$$

$$f_3 = 1+\epsilon^2\left[\frac{1}{2}+\frac{9}{2}\frac{k^2T}{g-2k^2T}+\frac{3}{16}\frac{k^2T}{g+k^2T}+\frac{9}{2}\frac{(k^2T)^2}{(g-2k^2T)^2}\right]. \tag{2.7}$$

For a pure gravity wave ($T = 0$), equation (2.4) becomes

$$\frac{\overline{F}_{ex}}{\omega} = \frac{1}{2}\frac{\epsilon^2}{k^2}c_0\,[c_0\,(\tfrac{1}{2}+\epsilon^2)+U(x,0)\,(1+\tfrac{1}{2}\epsilon^2)] = \text{const.}, \tag{2.8}$$

which is the same as the result of Crapper (1972) to $O(\epsilon^4)$‡.

3. Infinitesimal waves

Keeping the lowest order terms in (2.1) and (2.4), one obtains

$$\omega = c_0 k + U(x,0)\,k \tag{3.1}$$

and

$$\overline{F}_{ex} = \tfrac{1}{2}\epsilon^2\omega\frac{c_0}{k^2}\left[\tfrac{1}{2}c_0+\frac{Tk}{c_0}+U(x,0)\right] \tag{3.2}$$

† The ϵ used here is related to the surface displacement $\eta(x,t)$ by (Wehausen & Laitone 1960)

$$\eta(x,t) = \epsilon\eta^{(1)}+\epsilon^2\eta^{(2)}+\epsilon^3\eta^{(3)}+\dots,$$

with $\eta^{(1)} = (1/k)\cos\psi$,

$$\eta^{(2)} = \frac{1}{2k}\frac{g+k^2T}{g-2k^2T}\cos 2\psi,$$

$$\eta^{(3)} = \frac{1}{k}\left[\frac{1}{8}+\frac{3}{16}\frac{k^2T(5g+2k^2T)}{(g+k^2T)\,(g-2k^2T)}\right]\cos\psi+\frac{3}{16k}\frac{2g^2-gk^2T-30(k^2T)^2}{(g-2k^2T)\,(g-3k^2T)}\cos 3\psi;$$

ψ is the phase of the surface wave.

‡ The relation between ϵ and z in Crapper's paper is $z = 1+\epsilon^2+\frac{3}{4}\epsilon^4$, which follows from a calculation of the gravity wave dispersion relation to $O(\epsilon^4)$.

as the conserved quantities for infinitesimal waves. Longuet-Higgins & Stewart (1961) have derived this result for gravity waves ($T = 0$). The quantity

$$\tfrac{1}{2}c_0 + (Tk/c_0) + U(x, 0) \tag{3.3}$$

is the group velocity for gravity–capillary waves on a variable surface current $U(x, 0)$. By subtracting the kinetic and potential energy density of the current from the mean total energy density, one obtains

$$\tfrac{1}{2}\epsilon^2\omega(c_0/k^2) \tag{3.4}$$

as the mean *excess* energy density. Consequently, (3.2) becomes the energy theorem of Rayleigh and Reynolds (Rayleigh 1877; Reynolds 1877) applied to a wave on a variable current [mean excess energy flux = mean excess energy density × group velocity].

For a pure capillary wave (3.1) and (3.2) become

$$(Tk^3)^{\frac{1}{2}} - Sk + \delta Uk = (Tk_0^3)^{\frac{1}{2}} - Sk_0 \tag{3.5}$$

and

$$\epsilon^2 k^{-\frac{3}{2}}[\tfrac{3}{2}(Tk)^{\frac{1}{2}} - S + \delta U] = \epsilon_0^2 k_0^{-\frac{3}{2}}[\tfrac{3}{2}(Tk_0)^{\frac{1}{2}} - S]; \tag{3.6}$$

ϵ_0 and k_0 are the values of the wave slope and wavenumber where the current is $-S$, and δU is the variation of U about $-S$. For $S > (Tk_0)^{\frac{1}{2}}$ a critical current variation

$$\delta U_R = S - 3\{T/4[Sk_0 - (Tk_0^3)^{\frac{1}{2}}]\}^{\frac{1}{3}} \tag{3.7}$$

implies that $\epsilon \to \infty$, and k becomes

$$k_R = 2^{\frac{2}{3}}T^{-\frac{1}{3}}[Sk_0 - (Tk_0^3)^{\frac{1}{2}}]^{\frac{2}{3}}. \tag{3.8}$$

Since $\delta U_R > 0$, pure infinitesimal capillary waves are enhanced without bound by a positive current variation in contrast to pure infinitesimal gravity waves, the unlimited enhancement of which requires a negative or adverse current variation

4. Comparison of nonlinear and infinitesimal solutions

The differences between the nonlinear or finite amplitude solution (2.1)–(2.7) and the infinitesimal wave solution (3.1)–(3.2) are illustrated by figures 1 and 2, which show the wave-slope and wavenumber changes as a function of current variation for a 20 cm ($k = 0{\cdot}314\ \mathrm{cm}^{-1}$) gravity wave and a 1 cm ($k = 6{\cdot}28\ \mathrm{cm}^{-1}$) capillary wave, both of which have an initial slope of 0·1 on a current with $S = 25$ cm/s. The infinitesimal gravity wave is amplified and compressed – its wavenumber increases – as it encounters a negative or adverse current variation. At a current variation $\delta U = -0{\cdot}401$ cm/s, corresponding to zero group velocity, this wave can progress no further, and its wave slope becomes arbitrarily large. The wavenumber at this point is finite and equal to $0{\cdot}401\ \mathrm{cm}^{-1}$. The finite amplitude gravity wave propagates beyond $\delta U = -0{\cdot}401$ cm/s with a decreasing rate of slope increase as the effect of finite amplitude becomes more pronounced. The capillary wave is amplified and expanded – its wavenumber decreases – when it encounters a positive current variation. The infinitesimal capillary wave can

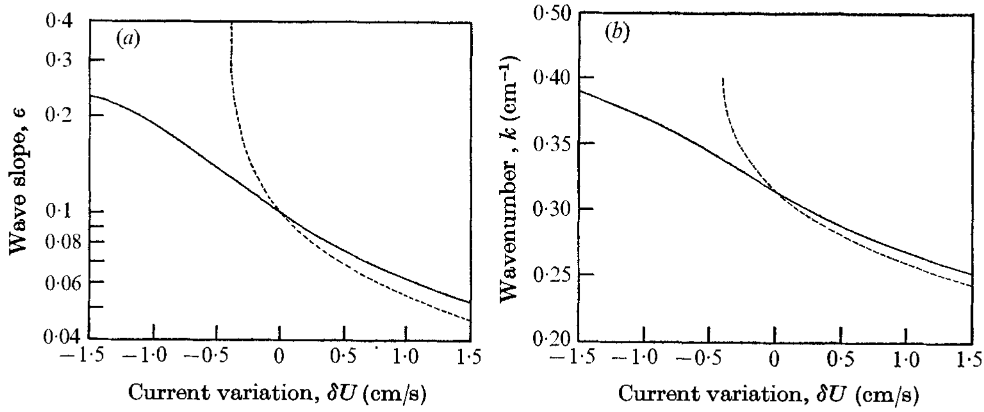

FIGURE 1. (*a*) Wave slope and (*b*) wavenumber as a function of current variation for an incident wave of $\epsilon = 0{\cdot}1$ and $k = 0{\cdot}134\,\text{cm}^{-1}$ ($\lambda = 20\,\text{cm}$) on a current of $S = 25\,\text{cm/s}^{-1}$ ($g = 981\,\text{cms}^{-2}$, $T = 72\,\text{dyne}\,\text{cm}^2\,\text{gm}^{-1}$). ---, infinitesimal wave theory; ——, finite amplitude theory.

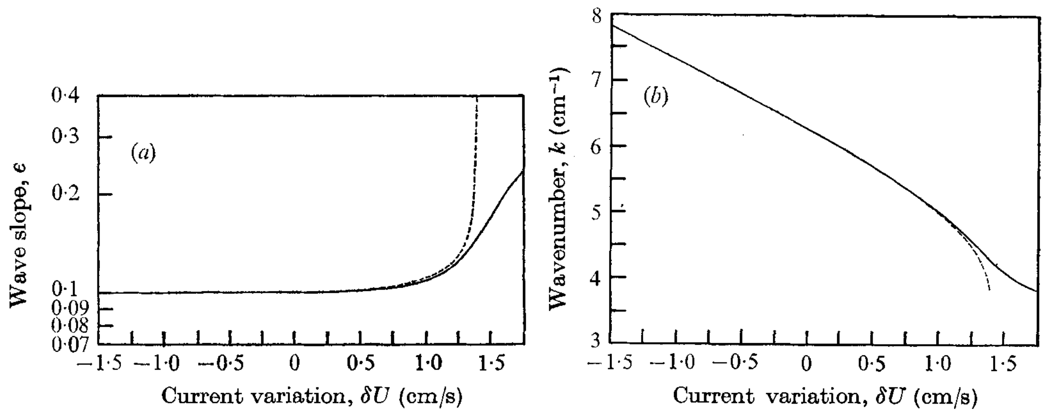

FIGURE 2. (*a*) Wave slope and (*b*) wavenumber as a function of current variation for an incident wave of $\epsilon = 0{\cdot}1$ and $k = 6{\cdot}28\,\text{cm}^{-1}$ ($\lambda = 1\,\text{cm}$) on a current of $S = 25\,\text{cm/s}$ ($g = 981\,\text{cms}^{-2}$, $T = 72\,\text{dyne}\,\text{cm}^2\,\text{gm}^{-1}$). ———, infinitesimal wave theory; ——, finite amplitude theory.

progress no further than $\delta U = 1.39$ cm/s, where its wave slope becomes arbitrarily large and its wavenumber reaches $3{\cdot}87\,\text{cm}^{-1}$. The finite amplitude capillary wave progresses beyond this point. A comparison of figures 2 (*a*) and (*b*) with figures 1 (*a*) and (*b*) indicates that the effect of finite amplitude is much less pronounced for capillary waves than for gravity waves. This conclusion can also be reached by examining (2.1)–(2.7) in the $g = 0$ and $T = 0$ limits.

Gargett & Hughes (1972), in an analysis based on infinitesimal gravity waves, have concluded that the weak surface current induced by an internal wave can create a 'barrier' to surface waves. Figures 1 (*a*) and (*b*) show that finite amplitude effects remove this 'wave barrier' and can reduce the amplitude enhancement considerably below that which would be predicted by an infinitesimal wave analysis.

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REFERENCES

CRAPPER, G. D. 1972 *J. Fluid Mech.* **52**, 713.

DEFANT, A. 1960 *Physical Oceanography*, vol. II. Pergamon.

GARGETT, A. E. & HUGHES, B. A. 1972 *J. Fluid Mech.* **52**, 179.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

LANDAU, L. D. & LIFSCHITZ, E. M. 1959 *Fluid Mechanics*. Pergamon.

LIGHTHILL, M. J. 1967 *Proc. Roy. Soc.* A **299**, 28.

LONGUET-HIGGINS, M. S. & STEWART, R. W. 1961 *J. Fluid Mech.* **10**, 529.

RAYLEIGH, LORD 1877 *Proc. Lond. Math. Soc.* **9**, 21.

REYNOLDS, O. 1877 *Nature*, **16**, 343.

WEHAUSEN, J. W. & LAITONE, E. V. 1960 Surface waves. *Handbuch der Physik*, vol. 9, *Fluid Mechanics*, III. Springer.

WHITHAM, G. B. 1962 *J. Fluid Mech.* **12**, 135.

WHITHAM, G. B. 1965*a* *Proc. Roy. Soc.* A **283**, 238.

WHITHAM, G. B. 1965*b* *J. Fluid Mech.* **22**, 273.